Lagrangian and Hamiltonian mechanics

* All that's needed, write down L or H and the eqns of motion

$$L(q,\dot{q},t) = \overset{\text{kinetic}}{T} - \overset{\text{potential}}{U}$$

Euler-Lagrange Eqns: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$ (one per q)

Ex: particle moving in 1D potential $U(x)$

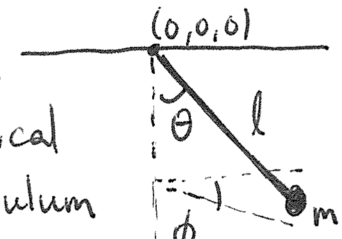
$$L = \frac{1}{2}m\dot{x}^2 - U(x) \qquad \frac{\partial L}{\partial \dot{x}} = \cancel{2}(\tfrac{1}{\cancel{2}}m)\dot{x} \qquad \frac{\partial L}{\partial x} = -U'(x) = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow \underline{m\ddot{x} = F}$$

if L does not depend on q, $\frac{\partial L}{\partial q} = 0 \rightarrow \frac{\partial L}{\partial \dot{q}}$ is conserved!

$\partial L/\partial \dot{q}$ is called the "conjugate momentum" of q

Ex: spherical pendulum



Lagrangian? Do it in cartesian first

$x = l\sin\theta\cos\phi$

$y = l\sin\theta\sin\phi$

$z = -l\cos\theta$

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$\dot{x} = l\left[\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi}\right]$

$\dot{y} = l\left[\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi}\right]$

$\dot{z} = +l\sin\theta\,\dot{\theta}$

$$\dot{x}^2+\dot{y}^2 = l^2\left[\cos^2\theta\underline{\cos^2\phi}\,\dot{\theta}^2 + \sin^2\theta\,\underline{\sin^2\phi\,\dot{\phi}^2} - \cancel{2\sin\theta\sin\phi\cos\theta\cos\phi\,\dot{\theta}\dot{\phi}} + \cos^2\theta\,\underline{\sin^2\phi\,\dot{\theta}^2} + \sin^2\theta\,\underline{\cos^2\phi}\,\dot{\phi}^2 + \cancel{2\cos\theta\sin\phi\sin\theta\cos\phi\,\dot{\theta}\dot{\phi}}\right]$$

$$= l^2\left[\cos^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right]$$

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = l^2\left[\underline{\cos^2\theta}\,\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 + \underline{\sin^2\theta}\,\dot{\theta}^2\right] = l^2\left[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right]$$

$$U = mgh = mgz = -mgl\cos\theta \rightarrow \underline{L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mgl\cos\theta}$$

Conserved quantity? i.e., $\frac{\partial L}{\partial q} = 0$ for which coordinate?

L has θ, but not ϕ $\rightarrow$ $\partial L / \partial \dot\phi$ is conserved

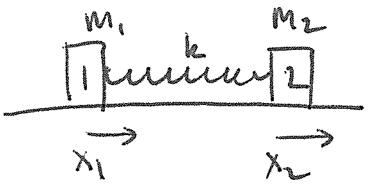
$$\frac{\partial L}{\partial \dot\phi} = \frac{1}{\cancel{2}} m l^2 (\cancel{2} \dot\phi \sin^2\theta) = \underline{m l^2 \sin^2\theta \, \dot\phi}$$

What if pendulum is confined to the plane $\phi = 0$? 3D pendulum $\rightarrow$ 1D

$$\dot\phi = 0 \rightarrow L = \tfrac{1}{2} m l^2 \dot\theta^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\partial L/\partial\dot\theta\right) = \partial L/\partial\theta \rightarrow \frac{d}{dt}\left(\cancel{m} l^{\cancel{2}} \dot\theta\right) = -\cancel{m} g \cancel{l} \sin\theta \rightarrow \underline{\ddot\theta = -g/l \sin\theta}$$

Ex:



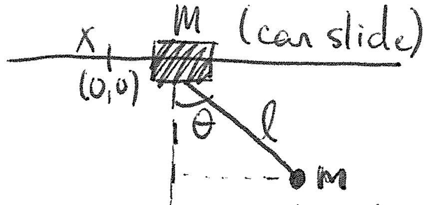
$$T = \tfrac{1}{2} m_1 \dot x_1^2 + \tfrac{1}{2} m_2 \dot x_2^2$$

$$U = \tfrac{1}{2} k \Delta x^2 = \tfrac{1}{2} k (x_2 - x_1)^2$$

$$L = \tfrac{1}{2}\left[m_1 \dot x_1^2 + m_2 \dot x_2^2\right] - \tfrac{1}{2} k (x_2 - x_1)^2$$

$$\frac{d}{dt}(\partial L/\partial \dot x_1) = d/dt\left(\tfrac{1}{\cancel{2}} \cancel{2} m_1 \dot x_1\right) = \underline{m_1 \ddot x_1} = \partial L/\partial x_1 = -k(x_2 - x_1)\cdot(-1) = \underline{k(x_2 - x_1)}$$

Ex:



$(x + l\sin\theta, -l\cos\theta)$

$\hookrightarrow (\dot x + l\cos\theta\,\dot\theta,\ l\sin\theta\,\dot\theta)$

$$T = \tfrac{1}{2} M \dot x^2 + \tfrac{1}{2} m\left(\dot x^2 + l^2 \underline{\cos^2\theta}\,\dot\theta^2 + 2l\cos\theta\,\dot x\dot\theta + l^2 \underline{\sin^2\theta}\,\dot\theta^2\right)$$

$$= \tfrac{1}{2}(M+m)\dot x^2 + \tfrac{1}{2} m\left(l^2\dot\theta^2 + 2l\cos\theta\,\dot x\dot\theta\right)$$

$$U = mgh = mgy = -mgl\cos\theta$$

$$L = T - U = \tfrac{1}{2}(M+m)\dot x^2 + \tfrac{1}{2} m l^2 \dot\theta^2 + ml\cos\theta\,\dot x\dot\theta + mgl\cos\theta$$

conserved quantity? $\partial L/\partial x = 0 \rightarrow \partial L/\partial \dot x = \left[(M+m)\dot x + ml\cos\theta\,\dot\theta\right]$ is conserved.

Eqn of motion?

$$d/dt(\partial L/\partial\dot\theta) = d/dt(ml^2\dot\theta) = \underline{\cancel{m} l^{\cancel{2}} \ddot\theta} = \partial L/\partial\theta = \underline{-\cancel{m}\cancel{l}\sin\theta\,\dot x\dot\theta - \cancel{m}g\cancel{l}\sin\theta}$$

$$\ddot\theta = \frac{-1}{l}\sin\theta\,\dot x\dot\theta - \underbrace{g/l\sin\theta}_{\text{normal pendulum}}$$

$$(M+m)\ddot x + ml\left[\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\right] \doteq 0$$

Hamiltonian

$$H(p,q) = \sum_i p_i \dot{q}_i - L \quad \text{(formal)} \qquad p_i = \partial L / \partial \dot{q}_i \ \text{(conjugate momentum!)}$$

solve to get $\dot{q}_i(q_i, p)$

Useless!

$H = T + U$ as long as $U \neq U(\dot{q}_i)$ and $U \neq U(t)$

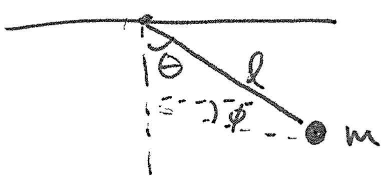
~~EM:~~ Hamilton's Eqns of Motion: $\begin{cases} \dot{p} = -\partial H/\partial q \\ \dot{q} = \partial H/\partial p \end{cases}$

Ex: 1D particle moving in potential

$$T = \frac{p^2}{2m} \rightarrow H = p^2/2m + U(x) \quad \dot{p} = \frac{-\partial H}{\partial \cancel{q}\, x} = -U'(x) \quad \dot{x}\cancel{\dot{q}} = \frac{\partial H}{\partial p} = p/m$$

in other words $\frac{dp}{dt} = F$ and $v = p/m$ (Use this to get signs right!)

Ex:



need momenta (p_θ, p_ϕ) $\quad U \neq U(\dot{q})!$

From L, $p_\theta = \partial L/\partial \dot{\theta} = \partial T/\partial \dot{\theta} = (l^2 \cdot 2\dot{\theta}) \cdot \frac{m}{2}$

$p_\phi = \partial L/\partial \dot{\phi} = (l^2 \sin^2\theta \cdot 2\dot{\phi}) \cdot m/2$

$p_\theta = ml^2\dot{\theta}$

$p_\phi = ml^2 \sin^2\theta\, \dot{\phi}$

$U = -mgl\cos\theta$

$$T = \frac{1}{2} ml^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) = \frac{1}{2}\left(\frac{p_\theta^2}{ml^2} + \frac{p_\phi^2}{ml^2 \sin^2\theta}\right)$$

$$H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2 \sin^2\theta} - mgl\cos\theta$$

Eqns of motion?

$$\begin{cases} \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{-p_\phi^2}{2ml^2}(-2\sin^{-3}\theta\cos\theta) - mgl\sin\theta \\ \dot{p}_\phi = -\partial H/\partial \phi = 0 \quad \text{(conserved)} \\ \dot{\theta} = \partial H/\partial p_\theta = \frac{p_\theta}{ml^2} \\ \dot{\phi} = \frac{p_\phi}{ml^2 \sin^2\theta} \end{cases}$$