Waves and Optics

A wave is a disturbance that propagates in time; it satisfies the equation 
\[ \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2} \] (1D)

Traveling waves solutions include any function \( f(x \pm vt) \). This leads to the principle of superposition: for any two solutions \( f(x,t) \) and \( g(x,t) \), \( f+g \) is also a solution.

Standing waves look like \( f(x,t) = A(x) B(t) \)
For example, \( f(x,t) = \cos x \cos vt = \frac{1}{2} (\cos (x+vt) + \cos (x-vt)) \)

GRE 1) Sound waves in air can be described by the equation 
\[ \frac{\partial^2 p}{\partial x^2} = K^2 \frac{\partial^2 p}{\partial t^2} \] where \( p \) is the pressure deviation and \( K \) is a constant. The speed of sound is
A) \( \frac{1}{K^2} \) B) \( 1/K \) C) \( \sqrt{K} \) D) \( K \) E) \( K^2 \)

GRE 3) Let \( f(x,t) \) and \( g(x,t) \) be two traveling waves solutions to the homogeneous wave equation. Which of the following are true?
I. \( f+g \) solves the wave equation
II. \( 2f-3g \) solves it
III. \( fg \) solves the wave equation
A) I B) II C) III D) I and III E) II and III

Solutions to the wave equation can all be written as linear combs of
\[ f(x,t) = A \cos (kx-\omega t+\delta) \] (thanks to Fourier)

\( A \) is amplitude, \( k \) : wave number, \( \omega \) : angular frequency, \( \delta \) : phase
\[ \lambda = \frac{2\pi}{k} \quad T = \frac{2\pi}{\omega} \quad \omega = 2\pi f \]
\[ \omega = \nu k \rightarrow \text{dispersion relation (linear case)} \]
phase velocity: \( \frac{w}{k} \) (same in linear case, like light in vacuum)

group velocity: \( \frac{d \omega}{dk} \leq \text{speed of information} \) (phase > \( c \) is possible)

Because \( \omega \) is a function of \( k \), it is wavelength dependent.

(Note: \( k \) is dispersed in prism, for example.

wave packet

GRE#16 from test: Two identical sine waves travel in opposite directions in a 15-m wire, producing a standing wave. The traveling waves have a speed of 12 m/s and the standing wave has six nodes, including those at the two ends. What is the wavelength and frequency?

\[
\begin{align*}
\lambda &= 2 \cdot 3 = 6 \text{ m} \\
v &= \frac{w}{k} = \frac{12 \text{ m/s}}{5} = \frac{24 \text{ Hz}}{5} = 4.8 \text{ Hz} \\
\end{align*}
\]

Interference

Consider \( f(x,t) = A\cos(kx - wt) \) \\
\( g(x,t) = A\cos(kx - wt + \pi) \) \\
\( f + g = 0 \) \\
(destructive)

constructive: phase difference of \( 2\pi n \) \\
destructive: " " " " \( (2n+1)\pi \)

double-slit interference

\[
\begin{align*}
\Delta x &\propto d \sin \theta \\
f(x_{net}) = A\cos(k(x+\Delta x) - wt) = A\cos(kx - wt + k\Delta x) \\
\delta &= k\Delta x \text{ (phase shift)}
\end{align*}
\]

sine, monochromatic source, two slits act as two point sources

\[
\Delta x \propto d \sin \theta
\]
maxima: \( \sin \theta = m \lambda \) (constructive) remember! \( \theta = 0 \to \max \)

minima: \( \sin \theta = (m + \frac{1}{2}) \lambda \) (destructive) so \( m \), not \( m + \frac{1}{2} \)

GRE #8) Monochromatic light (\( \lambda \)) is directed at two slits w/spacing \( \ell \).
If the same light is directed at a different two-slit arrangement with separation \( \ell' \), the position of the 3rd interference minimum corresponds to the position of the old 2nd interference max. after the central maximum. What is \( \ell' \) in terms of \( \ell \) and \( \lambda \)?

A) \( 4\ell' / 5 \)  B) \( 4\lambda^2 / 5\ell \)  C) \( \ell \)  D) \( 5\ell' / 4 \)  E) \( 5\ell^2 / 4\lambda \)

3rd min \( \to m = 2 \)  \( \ell' \sin \theta = \frac{3}{2} \lambda \) \( \sin \theta = \frac{3\lambda}{2\ell'} \) \( \ell' = \frac{5}{4} \ell \)

2nd max after central \( \to m = 2 \)  \( \ell' \sin \theta = 2\lambda \) \( \sin \theta = \frac{2\lambda}{\ell'} \)

Single-slit diffraction

\[ \begin{array}{c}
\text{leads to Rayleigh criterion}
\end{array} \]

for resolution of two distant point sources:

\( A \sin \theta = 1.22 \lambda \) or \( \Delta \ell = \frac{1.22 \lambda L}{a} \)

If \( \theta < \theta_{\text{crit}} \), maxima of two sources overlap, blurring the image

GRE #14 (test): Observer looks through slit of width \( 5 \times 10^{-4} \) m at two lanterns 1 km away. The lanterns emit light of wavelength \( 5 \times 10^{-7} \) m. The min. separation at which they can be resolved is most nearly

A) 0.01 m  B) 0.1 m  C) 1 m  D) 10 m  E) 100 m

\( \Delta \ell = \frac{1.22 \lambda L}{a} = 1.22 \cdot \frac{5 \times 10^{-7} m \cdot 10^3 m}{5 \times 10^{-4} m} = 1.22 m \)
Geometric optics

reflection: $\Theta_i = \Theta_r$ (incidence angle equals angle of reflected angle)

refraction: $n_1 \sin \Theta_1 = n_2 \sin \Theta_2$ (Snell's Law)

apply to any and all wave phenomena, e.g., sound, water, etc.

*angle is w/ respect to the normal

RE #11) A person at the bottom of a pool looking up at the sky observes the sun at an angle $\Theta$ from the horizon. What is the true angle of the sun, in terms of the index of refraction, of the water?

- $n_1 \sin \Theta_1 = n_2 \sin \Theta_2$
- $\sin (90 - \Theta_1) = n_2 \sin (90 - \Theta)$
- $\Theta' = \cos^{-1} (n_2 \cos \Theta)$

A) $\cos^{-1} (n \cos \Theta)$  B) $\sin^{-1} (n \cos \Theta)$  C) $\sin^{-1} (n \sin \Theta)$

D) $\cos^{-1} (\cos \Theta / n)$  E) $\sin^{-1} (\sin \Theta / n)$

Lenses and Mirrors

$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$  \hspace{1cm} f = focal length

s = object  \hspace{1cm} s' = image

for mirrors, $f = R/2$ (radius of curvature)

convex lenses  \hspace{1cm} concave mirrors  \hspace{1cm} $f > 0$

concave lenses  \hspace{1cm} convex mirrors  \hspace{1cm} $f < 0$
Distances $s$ and $s'$ are positive if on the same side as the light rays, negative if opposite. ($s$ incoming, $s'$ outgoing)

$s' > 0 \Rightarrow$ real image (can also happen in multi-lens setup)

$s' < 0 \Rightarrow$ virtual image

Magnification $M = -\frac{s'}{s}$ (sign determines if upright or inverted)

**GRE #15 (test)**

Two thin converging lenses A+B, each with focal length 6 cm, are 10 cm apart.

If an object is placed 10 cm to the left of A, the final image is where?

**GRE #14**

Converging lens of $f=6$ cm is 12 cm to the left of concave mirror of $R=6$ cm. An object is placed 3 cm to the right of the lens. How many real and virtual images are formed by the lens?

A) 1 real  B) 1 virtual  C) 2 real  D) 2 virtual  E) 1 each

Light can hit the object and go two directions, right or left.

For left, it goes through the lens only.

Case #1: $\frac{f}{s} = \frac{6}{3} \Rightarrow s' = (\frac{1}{6} - \frac{1}{3})^{-1} = -6$ cm virtual

(Above note object is inside focal length, meaning image is virtual.)
Case #2: light goes right, to mirror, and back through lens
mirror: \[ s' = \left( \frac{1}{3} - \frac{1}{4} \right)^{-1} = 4.5 \text{ cm} \] (real image)
lens: \[ s' = \left( \frac{1}{6} - \frac{1}{75} \right)^{-1} = 30 \text{ cm} \] (real image)

Lenses: 
\[
\frac{1}{f} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]
for thin lens

\[
\rightarrow \quad \left( R > 0 \quad \rightarrow \quad R < 0 \right)
\]

for converging lens we know \( f > 0 \) always, so \( R_1 > 0, R_2 < 0 \).

Thin films
(no time)